Appendix GG

Transitions with *x*- and *y*-Polarized Light

We would now like to calculate transition probabilities for light polarized in the x- and y-directions. For this purpose, we introduce linear combinations of the x- and y-coordinates having simple transition integrals. We define $r_+ = x + iy$ and $r_- = x - iy$. Using Eq. (4.3) and Euler's formula, one may derive the following equations for r_+ and r_-

$$r_+ = r \sin\theta \, e^{i\phi}, \tag{GG.1}$$

$$r_- = r \sin\theta \, e^{-i\phi}. \tag{GG.2}$$

The significance of the variables r_+ and r_- may be understood in physical terms. If we were to multiply r_+ by the time dependence $e^{-i\omega t}$ corresponding to an oscillating electric field, we would obtain

$$r_+(t) = r \sin\theta \, e^{i(\phi - \omega t)}. \tag{GG.3}$$

$r_+(t)$ has the same dependence upon the polar angle and time as the components of a polarization vector rotating about the z-axis. We may thus associate the variable r_+ with circularly polarized light. Similarly, r_- may be associated with circularly polarized light for which the polarization vector rotates about the z-axis in the opposite direction. We now use the variables r_+ and r_- to evaluate transition integrals for x- and y-polarized light.

Example GG.1

Again for the $2p \to 1s$ transition of the hydrogen atom, calculate the transition integrals for x- and y-polarized light.

Solution

For the transition 2p-1 to 1s0, the integral of r_+ is

$$\int \phi^*_{1s0} \, r_+ \, \phi_{2p-1} \, dV = R_i \int_0^\pi \frac{1}{\sqrt{2}} \cdot \sin\theta \cdot \frac{\sqrt{3}}{2} \sin\theta \sin\theta \, d\theta \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} \cdot e^{i\phi} \cdot \frac{1}{\sqrt{2\pi}} e^{-i\phi} \, d\phi, \tag{GG.4}$$

where R_i is the radial integral given by Eq. (4.25). The integration over ϕ gives 1, and the θ integration can be written

$$\frac{1}{2}\sqrt{\frac{3}{2}} \int_0^\pi \sin^3\theta \, d\theta = \frac{1}{2}\sqrt{\frac{3}{2}} \cdot \frac{4}{3} = \sqrt{\frac{2}{3}}. \tag{GG.5}$$

The transition integral for the operator r_+ is thus

$$\int \phi^*_{1s0} \, r_+ \, \phi_{2p-1} \, dV = \sqrt{\frac{2}{3}} R_i \tag{GG.6}$$

The integrals of r_+ for the transition $2p0 \to 1s0$ and $2p+1 \to 1s0$ may be shown to be equal to zero. The integral of r_- for the transition $2p+1$ to $1s0$ can be shown to be

$$\int \phi^*_{1s0} \, r_- \, \phi_{2p+1} \, dV = -\sqrt{\frac{2}{3}} R_i, \tag{GG.7}$$

and the other integrals of r_- are zero. The defining equations for r_+ and r_- can be solved for x and y to obtain

$$x = \frac{1}{2}(r_+ + r_-) \tag{GG.8}$$

$$y = \frac{1}{2i}(r_+ - r_-). \tag{GG.9}$$

These equations can be used to evaluate the transition integrals for x-and y-polarized light.

The transition rates for x- and y-polarized light can be calculated using Eqs. (4.20) and (4.21) with x and y in place of z. The calculation of these transition rates are left as an exercise (see Problem 18 of Chapter 4).